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Nonlinear electrodynamics of electrons in a quasi-one-dimensional ballistic ring

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Abstract. We consider ballistic electron motion in a quasi-one-dimensional ring under the uniform high-frequency electric field induced by an electromagnetic field. The electron satisfies a nonlinear equation of motion which is formally identical to that for a pendulum with a vibrating suspension point. The averaging method of Kapitza is used. The electromagnetic emission spectrum is calculated. The spectrum consists of low-frequency radiation, scattered radiation at the incident radiation frequency and combination scattered radiation; the intensities and frequencies of all components depend nonlinearly on the incident radiation frequency. At a certain value of that intensity the spontaneous symmetry breakdown occurs. As a result, the system acquires some static electric dipole moment.

There are many papers devoted to the effects in quasi-one-dimensional rings. Various quantum phenomena are mainly considered, such as quantum confinement, the Aharonov–Bohm effect, Coulomb blockade, etc. In this paper we pay attention to the possibility of some interesting *classical* effects in quasi-one-dimensional ballistic rings under applied fields.

Consider a ballistic ring (the electron mean free path exceeds the ring circumference) of radius R under a uniform high-frequency electric field induced by a plane polarized electromagnetic wave that propagates along the normal to the ring plane. In the spatially uniform electric field of the wave, $\vec{F}(t) = \vec{F}_0 \cos\omega t$ affects electrons in the ring. However, electrons can only move along the ring circumference, so that the driving force $\vec{f}(\phi, t) = \{eF_0 \cos\omega t \cos\phi, eF_0 \cos\omega t \sin\phi\}$ (ϕ is the angular coordinate counting of the line parallel to the wave field) depends nonlinearly on the electron position in the ring, i.e. a 'geometrical nonlinearity' takes place. Electron motion in the ring is described by the nonlinear equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + \omega_0^2 \cos\omega t \sin\phi = 0 \qquad \omega_0 = \sqrt{\frac{eF_0}{mR}} \qquad eF_0 > 0 \tag{1}$$

($e < 0$ and m are the electron charge and effective mass, respectively). A mechanical analogue of the problem is a pendulum in the horizontal plane whose vertical suspension axis vibrates along a horizontal line; such a system was investigated by Kapitza half a century ago [1]. Note also that equation (1) has a form of (one-dimensional) equation of electron motion in a non-homogeneous electric field of the standing electromagnetic wave. For the case of a highly oscillating field this equation was investigated in [2, 3] by Kapitza's averaging method [1, 4].

It follows from the nonlinearity of equation (1) that electrons in the ballistic ring form a system with nonlinear electrodynamical (optical) properties even if the ring is electrodynamically linear *per se*. We consider some of the nonlinear effects below.

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In this paper we consider the case of the field $\vec{F}(t)$ highly oscillating with a high frequency ω, and carry out the averaging of the solution of equation (1) over the period $2\pi/\omega$ by Kapitza's averaging method. The high-frequency condition is $\omega \gg 1/T_0$, where T_0 is the typical time of slow dynamics of averaged motion.

Following [1–3] we present the function $\phi(t)$ in the form

$$\phi(t) = \bar{\phi}(t) + \tilde{\phi}(t) \tag{2}$$

where $\bar{\phi}(t)$ is a slowly changing (with respect to time) function and $\tilde{\phi}(t)$ is highly oscillating (with frequency ω). Substituting (2) into (1) and expanding in a series of $\tilde{\phi}$ we get

$$\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\tilde{\phi}}{\mathrm{d}t^2} + \omega_0^2 \sin\bar{\phi}\cos\omega t + \omega_0^2\tilde{\phi}\cos\bar{\phi}\cos\omega t = 0. \tag{3}$$

Here the last term is small because of the smallness of $\tilde{\phi}(t)$ (as far as the second derivative $\mathrm{d}^2\tilde{\phi}/\mathrm{d}t^2$ is concerned, it is proportional to the large value ω^2 and is therefore not small). Thus, from equation (3) we get the 'fast' component of motion

$$\tilde{\phi} = \left(\frac{\omega_0}{\omega}\right)^2 \sin\bar{\phi}\cos\omega t. \tag{4}$$

Substituting (4) into (3) and averaging over time (considering that the functions $\sin\bar{\phi}$ and $\cos\bar{\phi}$ are constant), we get

$$\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}t^2} + \gamma^2 \sin\bar{\phi}\cos\bar{\phi} = 0 \qquad \left(\gamma = \frac{\omega_0^2}{\sqrt{2}\omega} = \frac{eF_0^2}{\sqrt{2}mR\omega} \ll \omega_0 \ll \omega\right). \tag{5}$$

Note that the second term in (5) is the analogue of the so-called Miller force [2, 3].

With initial conditions $\bar{\phi}(0) = 0$ and $\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}t}(0) = \Omega \equiv \frac{1}{R}\sqrt{\frac{2E}{m}}$ (E is the electron energy), equation (5) has the solution

$$\sin\bar{\phi}(t) = \begin{cases} \alpha\,\mathrm{sn}(\gamma t, \alpha) & E_0 > E \\ \mathrm{sn}(\alpha\gamma t, \alpha^{-1}) & E_0 < E \\ \tanh(\gamma t) & E_0 = E \end{cases} \tag{6}$$

where $\alpha = \Omega/\gamma = \sqrt{E/E_0}$, $E_0 = e^2F_0^2/4m\omega^2$ and $\mathrm{sn}(x, k)$ is the Jacobi elliptic sine with module k.

The low-frequency electric dipole moment of the ring that contains an electron and smeared compensating charge (the jellium model) has the form

$$\vec{p}(t) = eR\{\cos\bar{\phi}, \sin\bar{\phi}\} = \begin{cases} eR\{\pm\mathrm{dn}(\gamma t, \alpha), \alpha\,\mathrm{sn}(\gamma t, \alpha)\} & E_0 > E \\ eR\{\pm\mathrm{cn}(\alpha\gamma t, \alpha^{-1}), \mathrm{sn}(\alpha\gamma t, \alpha^{-1})\} & E_0 < E \\ eR\{\mathrm{sech}(\gamma t), \tanh(\gamma t)\} & E_0 = E \end{cases} \tag{7}$$

where $\mathrm{cn}(x, k)$ and $\mathrm{dn}(x, k)$ are the elliptic cosine and delta amplitude with module k, respectively.

First we consider the electrostatic properties of the system by averaging over the low-frequency motion, i.e. we take the zeroth harmonic of the Fourier series for equations (7). We then have

$$\vec{p}_0 = eR\left\{\pm\frac{\pi}{2K(\alpha)}, 0\right\} \quad \text{at} \quad E_0 > E \qquad \text{and} \qquad \vec{p}_0 = 0 \quad \text{at} \quad E_0 < E \tag{8}$$

where $K(k)$ is the complete elliptic integral of the first kind with module k.

It follows from equation (8) that a bifurcation takes place at $E_0 = E$. As a result, a system acquires an electric dipole moment. The effect has a simple interpretation. In the presence of the high-frequency field, electron motion averaged over the fast oscillations occurs in an effective two-well potential with minima at $\phi = 0$ and $\phi = \pi$ and a barrier of height E_0 between them (cf [1]). At $E_0 < E$ the electron energy exceeds the height of the barrier between wells so that the electron now moves over the barrier. At $E_0 > E$ the electron falls into one of the wells and cannot escape, so that at $E_0 = E$ spontaneous symmetry breakdown takes place and a nonzero electric dipole moment appears (optical rectification). This effect is analogous to the appearance of spontaneous polarization at a ferroelectric second-order phase transition with dipole moment $\vec{p}_0$ as an order parameter and the electric field F_0 as a control parameter.

Now, consider the time-dependent response of the ring to an electromagnetic wave. It is known [5] that, in the scattering of the electromagnetic wave by a nonrelativistic free electron in the dipole approximation, the scattered radiation has the same frequency as the incident radiation. The above-mentioned 'geometrical nonlinearity' of the 'electron in the ballistic ring' system leads to the appearance of a low-frequency response at the frequency of electron oscillations, resulting in a two-well potential and higher harmonics, as well as a high-frequency response at the frequency of the external field and combination frequencies that are created from addition or subtraction of the external field frequency with the low-frequency harmonics.

The electron dipole radiation spectrum is related to the Fourier coefficients of the dipole moment $\vec{p}(t) = \sum_{n=0}^{\infty}(\vec{a}_n \cos n\nu t + \vec{b}_n \sin n\nu t)$ by the formula [5]

$$\bar{J}_n = \frac{1}{3c^3} n^4 \nu^4 (\vec{a}_n^2 + \vec{b}_n^2) \qquad (n = 1, 2, 3, \ldots) \tag{9}$$

where c is the velocity of light and $\bar{J}_n$ is the radiation intensity at the nth harmonic frequency.

Using the known formulae for the Fourier expansion of the Jacobi functions [6] we obtain from equations (7) and (9) the following results for the low-frequency part of the scattering spectrum.

The fundamental frequency of the low-frequency radiation is equal to

$$\nu = \begin{cases} \dfrac{\pi\gamma}{2K(\alpha)} & E_0 > E \\ \dfrac{\pi\gamma\alpha}{2K(\alpha^{-1})} & E_0 < E \end{cases} \tag{10}$$

and tends to 0 at $E \to E_0$.

At $E_0 > E$ the intensities of the odd and even harmonics of that frequency have the following forms, respectively:

$$\bar{J}_{(2n-1)\nu} = \frac{\pi^6 e^2 R^2 \gamma^4}{12c^3 K^6(\alpha)} \frac{(2n-1)^4 q^{2n-1}}{(1-q^{2n-1})^2} \tag{11}$$

$$\bar{J}_{2n\nu} = \frac{\pi^6 e^2 R^2 \gamma^4}{12c^3 K^6(\alpha)} \frac{(2n)^4 q^{2n}}{(1+q^{2n})^2}. \tag{12}$$

At $E_0 < E$ the even harmonics are absent and the intensities of the odd ones have the form

$$\bar{J}_{(2n-1)\nu} = \frac{\pi^6 e^2 R^2 \gamma^4 \alpha^6}{6c^3 K^6(\alpha^{-1})} \frac{(2n-1)^4 q^{2n-1}(1+q^{4n-2})}{(1-q^{4n-2})^2}. \tag{13}$$

In formulae (11)–(13) the value $q = q(\alpha)$ is defined by

$$q = \begin{cases} \exp\left(-\dfrac{\pi K'(\alpha)}{K(\alpha)}\right) & E_0 > E \\ \exp\left(-\dfrac{\pi K'(\alpha^{-1})}{K(\alpha^{-1})}\right) & E_0 < E \end{cases} \tag{14}$$

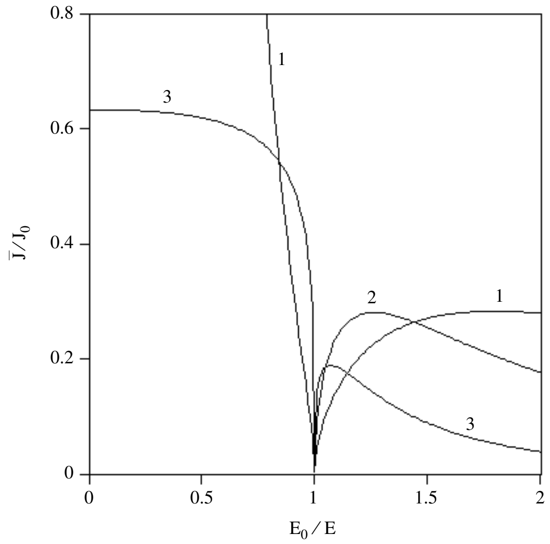

Figure 1. Dependence of the dipole emission intensity $\bar{J}_{s\nu}/J_0$ ($J_0 = e^2R^2\gamma^4/3c^3$) on the parameter $1/\alpha^2 = E_0/E$. The curves correspond to the numbers of the harmonics: (1) $s = 1$; (2) $s = 2$; (3) $s = 3$.

where $K'(k) \equiv K(k')$, $k' = \sqrt{1-k^2}$. Note that

$$\lim_{E\to E_0} q(\alpha) = 1 \qquad \lim_{E\to E_0} \bar{J}_{n\nu}(\alpha) = 0. \tag{15}$$

The dependence of the dipole emission intensity on the incident radiation intensity is shown in figure 1.

The even harmonics in the scattering spectrum at $E_0 > E$ have the same origin as the above-mentioned static dipole moment.

The evolution of the low-frequency dipole radiation of the electron in the ballistic ring with increase in the incident radiation intensity is described by equations (10)–(14) and has the following form.

At low intensities ($E_0 < E$) increasing the intensity leads to thickening of the discrete spectrum and an increase in the intensity of the odd higher harmonics. At $E_0 = E$ the spectrum becomes continuous. With further increases in the incident radiation intensity ($E_0 > E$) an opposite process occurs, i.e. thinning the discrete spectrum and decreasing intensity of the odd higher harmonics. In the limiting case $E \ll E_0$ the fundamental frequency coincides with the small oscillation frequency γ in one of the effective potential wells. The radiation intensity at this frequency is then equal to

$$J_\nu = J_\gamma = \frac{e^4 E F_0^2}{3c^3m^3R^2\omega^2}. \tag{16}$$

From equations (4) and (9) we obtain the spectral intensities for the high-frequency radiation. It is convenient to represent them in the form of the corresponding scattering cross sections, $\sigma = J/(cF_0^2/8\pi)$. The cross section at the incident radiation frequency is given by

$$\sigma_\omega = \begin{cases} \sigma_T\left[1 - \dfrac{E(\alpha)}{K(\alpha)}\right]^2 & E_0 > E \\ \sigma_T\cdot\alpha^4\left[1 - \dfrac{E(\alpha^{-1})}{K(\alpha^{-1})}\right]^2 & E_0 < E. \end{cases} \tag{17}$$

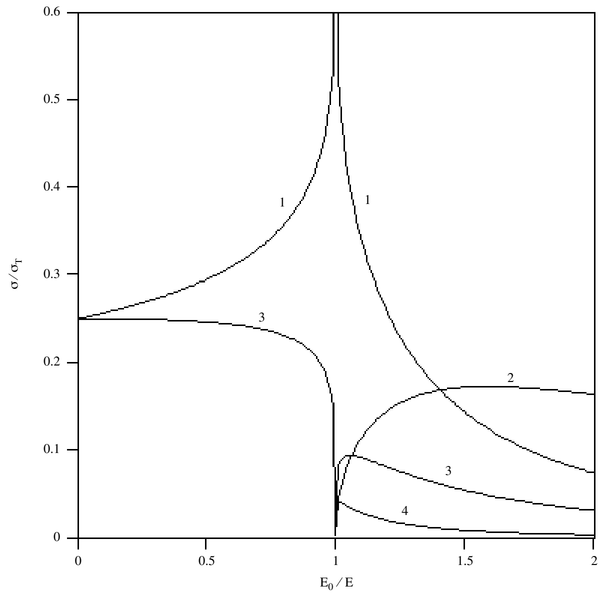

Figure 2. Dependence of the scattering cross section $\sigma_{\omega\pm s\nu}/\sigma_T$ on the parameter $1/\alpha^2 = E_0/E$. The curves correspond to the number of the harmonics: (1) $s = 0$; (2) $s = 1$; (3) $s = 2$; (4) $s = 3$.

Here $\sigma_T = 8\pi e^4/3m^2c^4$ is the Thomson scattering cross section and $E(k)$ is the complete elliptic integral of the second kind with module k.

At $E_0 > E$ the scattering cross sections at combination frequencies are

$$\sigma_{\omega\pm(2n-1)\nu} = \sigma_T \cdot \frac{\pi^4}{2K^4(\alpha)} \frac{(2n-1)^2 q^{2n-1}}{(1+q^{2n-1})^2} \tag{18}$$

$$\sigma_{\omega\pm 2n\nu} = \sigma_T \cdot \frac{\pi^4}{2K^4(\alpha)} \frac{(2n)^2 q^{2n}}{(1-q^{2n})^2}. \tag{19}$$

At $E_0 < E$ the odd harmonics are absent and the even ones are

$$\sigma_{\omega\pm 2n\nu} = \sigma_T \cdot \frac{\pi^4\alpha^4}{K^4(\alpha^{-1})} \frac{(2n)^2 q^{2n}(1+q^{4n})}{(1-q^{4n})^2}. \tag{20}$$

The intensity dependence of the scattering cross sections is shown in figure 2. Note that different points of the curves correspond to different combination frequencies $\omega \pm n\nu$, depending on α. Note that

$$\lim_{E\to E_0} \sigma_\omega(\alpha) = \sigma_T \qquad \lim_{E\to E_0} \sigma_{\omega\pm n\nu}(\alpha) = 0. \tag{21}$$

Thus the following phenomena arise under geometric nonlinearity conditions:

(1) low-frequency emission with frequencies and intensities dependent on the incident radiation intensity;
(2) nonlinear coherent scattering with intensity-dependent cross section;
(3) stimulated combination (Raman) scattering with the satellite line positions and intensities that depend on the incident radiation intensity in a nonlinear way.

The total scattering cross section is given by

$$\sigma = \begin{cases} \sigma_T \left[1 - \dfrac{E(\alpha)}{K(\alpha)}\right] & E_0 > E \\ \sigma_T \cdot \alpha^2 \left[1 - \dfrac{E(\alpha^{-1})}{K(\alpha^{-1})}\right] & E_0 < E. \end{cases} \tag{22}$$

Up to this point, we have assumed that there is one electron in the ring. If N electrons with energy E are injected pointwise into the ring then a factor N^2 needs to be introduced into equations (11)–(13) and (16)–(20) for the radiation intensity and spectra, respectively. If the electron gas in the ring is degenerate then the energy E has the meaning of the Fermi energy. The Coulomb interaction between electrons may be neglected, on the condition that $\gamma \gg \omega_p$, where ω_p is the plasma frequency for electrons in the ring (the calculation of ω_p will be published elsewhere).

Instead of the pointwise injection, the following contactless procedure may be proposed. First, without an electromagnetic field, a uniform constant electric field F is applied (e.g. by means of a parallel-plate capacitor) that is parallel to the future direction of the electromagnetic wave field and satisfies the condition $F \gg E/2eR$. Constant-field polarization of the electron gas in the ring ensures that the initial condition $\phi(0) = 0$ for all the electrons is fulfilled. Then the electromagnetic field is turned on, which corresponds to the required value of the parameter α, and the polarizing constant field is turned off.

If the electrons in the ring are nonmonoenergetic the spectral line broadening due to electron energy spread is expected.

At $E_0 > E$, N electrons are distributed between two potential wells of the effective potential. If N is odd then the electrons cannot divide equally and the ring has an electric dipole moment. At $E \ll E_0$ the moment is equal to eR. On the other hand, if N is even the dipole moment may or may not differ from zero. If one performs measurements on such a ring many times with the electromagnetic field being turned on and off, then the dipole moment averaged over many measurements vanishes and the standard deviation is equal to $\frac{1}{2}|e|R\sqrt{N}$. In addition, the ring has an electric quadrupole moment of the order of NeR^2. If the electrons are distributed equally between two wells under even N, so that the dipole moment is equal to zero, then the quadrupole moment tensor has components $D_{xx} = -D_{yy} = \frac{3}{2}NeR^2$, $D_{zz} = 0$ (the ring lies in the xy plane with the x axis parallel to the electric field).

The static effects in question may be classified as the single-electron phenomena in which the system behaviour depends on the electron number in the system and parity effects.

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